Αριθμητική Ανάλυση. No. 1 Date 18/11/2019

Μέθοδος Απαλοιφής Gauss

$$\begin{array}{ll} & 9x_1 + 6x_2 + 3x_3 = 0 \\ \frac{6}{9} = \frac{2}{3} & 6x_1 + 8x_2 + 4x_3 = 6 \\ \frac{3}{9} = \frac{1}{3} & 3x_1 + 4x_2 + 3x_3 = 2 \end{array} \Leftrightarrow \begin{array}{ll} & 9x_1 + 6x_2 + 3x_3 = 0 \\ & 4x_2 + 2x_3 = 6 \\ \frac{2}{4} = \frac{1}{2} & 2x_2 + 2x_3 = 2 \end{array} \Leftrightarrow$$

$$\Leftrightarrow \left.\begin{array}{r} 9x_1 + 6x_2 + 3x_3 = 0 \\ 4x_2 + 2x_3 = 6 \\ x_3 = -1 \end{array}\right\} \Leftrightarrow \begin{array}{l} 9x_1 = 0 - 6x_2 - 3x_3 = 6 \cdot 2 - 3(-1) = -9 \Leftrightarrow x_1 = -1. \\ 4x_2 = 6 - 2x_3 = 6 - 2(-1) = 8 \Leftrightarrow x_2 = 2. \\ x_3 = -1. \end{array}$$

$$\begin{array}{c|ccc|c} & \multicolumn{3}{c|}{\overbrace{}^{A}} & b \\ & 9 & 6 & 3 & 0 \\ \frac{2}{3} & 6 & 8 & 4 & 6 \\ \frac{1}{3} & 3 & 4 & 3 & 2 \\ \hline & & 4 & 2 & 6 \\ 1/2 & & 2 & 2 & 2 \\ & & & 1 & -1 \end{array}$$

$x^T = [-1 \quad 2 \quad -1]$

$Ax = b \quad A \in \mathbb{R}^{n,n} \quad x, b \in \mathbb{R}^n, \quad A^{(1)}x = b^{(1)}$

$$A^{(1)} = \begin{array}{c} \\ m_{21} \\ \vdots \\ m_{n1} \end{array} \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \dots & a_{2n}^{(1)} \\ \vdots & & & \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} \end{pmatrix}, \quad b = \begin{pmatrix} b_1^{(1)} \\ b_2^{(1)} \\ \vdots \\ b_n^{(1)} \end{pmatrix} \longrightarrow A^{(2)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ & \vdots & & \\ & a_{n2}^{(2)} & & a_{nn}^{(2)} \end{pmatrix},$$

$$b^{(2)} = \begin{pmatrix} b_1^{(1)} \\ b_2^{(2)} \\ \vdots \\ b_n^{(2)} \end{pmatrix} \to \dots \to A^{(r)} = \begin{array}{c} \\ \\ m_{r+1,r} \\ \vdots \\ m_{n,r} \end{array} \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1r}^{(1)} & \dots & a_{1n}^{(1)} \\ & a_{22}^{(2)} & \dots & a_{2r}^{(2)} & \dots & a_{2n}^{(2)} \\ & 0 & & a_{rr}^{(r)} & \dots & a_{rn}^{(r)} \\ & & & a_{nr}^{(r)} & \dots & a_{nn}^{(r)} \end{pmatrix}, \quad b^{r} = \begin{pmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_r^{(r)} \\ \vdots \\ b_n^{(r)} \end{pmatrix}$$

$m_{i} = a_{ij}^{(1)} / a_{jj}^{(1)}$

$a_{ij}^{(2)} = a_{ij}^{(1)} - m_{ij} a_{ij}^{(1)}, \quad i,j = 2,3,\dots,n \quad b_i^{(2)} = b_i^{(1)} - m_{ij} b_i^{(1)}$

$m_{ir} = a_{ir}^{(r)} / a_{rr}^{(r)}$

$a_{ij}^{(r+1)} = a_{ij}^{(r)} + m_{ir} a_{rj}^{(r)}, \ i,j = r+1, r+2, \dots, n$

$b_i^{(r+1)} = b_i^{(r)} - m_{ir} b_r^{(r)}, \ i = r+1, r+2, \dots n$

Διαιρέσεις:

$$\sum_{k=1}^{n-1} k = \frac{(n-1)n}{2} = \frac{n^2}{2} - \frac{n}{2}$$

Πολλαπλασιασμοί - Προσθέσεις:

$$\sum_{k=1}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6} = \frac{n^3}{3} + P_2(n) = O\left(\frac{n^3}{3}\right)$$

και $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$, για το διάνυσμα b.

$A^{(2)} = M_1 A^{(1)}, \quad A^{(r+1)} = M_r A_r$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -m_{21} & 1 & 0 & \dots & 0 \\ -m_{31} & 0 & 1 & \dots & 0 \\ -m_{n1} & 0 & 0 & \dots & 1 \end{pmatrix}, \quad M_r = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & -M_{r+1,r} & \\ & & & -M_{nr} & 1 \end{pmatrix}$$

$U = A^{(n)} = M_{n-1} A^{(n-1)} = M_{n-1} M_{n-2} A^{(n-2)} = \dots = M_{n-1} M_{n-2} \dots M_2 M_1 A^{(1)} =$
$= MA$

όπου M κάτω τριγωνικός με μονάδες στη διαγώνιο

$U = MA \Leftrightarrow A = M^{-1} U = L$

όπου L κάτω τριγωνικός με μονάδες στη διαγώνιο

$L = M^{-1} = (M_{n-1} M_{n-2} \dots M_2 M_1) = M_1^{-1} M_2^{-1} \dots M_{n-2}^{-1} \cdot M_{n-1}^{-1}.$

$$M_1^{-1} = \begin{bmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & 0 & 1 & \\ \vdots & \vdots & \vdots & \ddots \\ m_{n1} & 0 & 0 & 1 \end{bmatrix}, \quad M_2^{-1} = \begin{bmatrix} 1 & & & O \\ 0 & 1 & & \\ 0 & m_{32} & 1 & \\ \vdots & \vdots & \vdots & \ddots \\ 0 & m_{n2} & 0 & \end{bmatrix}$$

$$M_1^{-1} M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ m_{n1} & m_{n2} & 0 & \cdots & 1 \end{bmatrix}$$

Επαγωγικά έπεται ότι:

$$L = \begin{bmatrix} 1 & & & & O \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & & & \ddots & \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & 1 & \\ m_{n1} & m_{n2} & m_{n3} & m_{n,n-1} & 1 \end{bmatrix}$$

$$Ax = b \Leftrightarrow L\underbrace{Ux}_{y} = b \Leftrightarrow \begin{cases} Ly = b \\ Ux = y \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & & \\ 2/3 & 1 & \\ 1/3 & 1/2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 \\ 6 \\ 2 \end{bmatrix}}_{b}$$

$$y^t = [0 \ 6 \ 1]$$

$$\begin{array}{cccc} \overbrace{\qquad\qquad\qquad}^{u} & & & y \\ 9 & 6 & 3 & 0 \\ & 4 & 2 & 6 \\ & & 1 & -1 \end{array}$$

$$x^T = [-1 \ 2 \ -1]$$